MATHEMATICS

THE DUAL SPACE OF THE INVERSE LIMIT OF AN INVERSE LIMIT SYSTEM OF BOOLEAN ALGEBRAS

BY

PH. DWINGER

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Introduction

This paper is concerned with the dual space of the inverse limit of an inverse limit system of Boolean algebras. Direct and inverse limit systems of Boolean algebras were studied in [2]. It is known that the dual space of the direct limit of a direct limit system of Boolean algebras is the inverse limit space of the inverse limit system of the dual spaces [2]. However in the case of an inverse limit system of Boolean algebras the situation is much more complicated. This is caused by the fact that the direct limit set of the direct limit system of dual spaces does not have a natural topology which makes it into the dual space of the inverse limit algebra. In fact, the cardinal number of the direct limit set will in general be smaller than the cardinal number of the inverse limit algebra (cf. section 8, Example). It was shown by HAIMO [2] that if $\{B_\alpha, f_{\alpha\beta}\}$ is an inverse limit system of Boolean algebras over a directed set Γ whose inverse limit algebra is B_∞, then the direct limit set X^∞ of the direct limit system of dual spaces can be mapped by a one-one map onto a subset of the dual space X of B_∞, provided all the projection maps $\pi_\alpha\colon B_\infty \to B_\alpha$ are "onto". However the conclusion in [2] that this subset of X is a closed subset is false. In fact, this subset is always dense in X and it is in general a proper subset of X.

The main objective of this paper is to establish that X^∞ can be topologized in natural way such that X^∞ becomes a zero dimensional [1]) Hausdorff space, and such that the dual space X of B_∞ is the largest zero dimensional Hausdorff compactification of X^∞ (theorem 6). It should be noted that this result only holds under the condition that all projection maps $\pi_\alpha\colon B_\infty \to B_\alpha$ are "onto". Without any restrictions imposed on the π_α, the space X^∞ is still zero dimensional but need not be a Hausdorff space. We will show that in this general situation X is the largest zero dimensional Hausdorff compactification of a quotient space of X^∞ (theorem 5). As a special case we shall discuss inverse limit sequences of Boolean algebras (theorem 7).

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1) zero dimensional is used in the sense that the space has a base of open-and-closed sets.

1. *Definitions and preliminaries*

An *inverse limit system* of Boolean algebras over a directed set Γ [2]) is a collection of Boolean algebras B_α, $\alpha \in \Gamma$ and of homomorphic maps $f_{\alpha\beta} : B_\beta \to B_\alpha$ defined for all $\alpha, \beta \in \Gamma, \alpha \leqslant \beta$ such that (1) $f_{\alpha\alpha}$ is the identity map for every $\alpha \in \Gamma$; (2) $f_{\alpha\gamma} = f_{\alpha\beta} f_{\beta\gamma}$, whenever $\alpha \leqslant \beta \leqslant \gamma$, $\alpha, \beta, \gamma \in \Gamma$. Such an inverse limit system over Γ will be denoted by $\{B_\alpha, f_{\alpha\beta}\}$. The *inverse limit* B_∞ will be defined as a subalgebra of the direct product $\underset{\alpha \in \Gamma}{\times} B_\alpha$.

We recall that $\underset{\alpha \in \Gamma}{\times} B_\alpha$ is the set of all function x on Γ such that for every $\alpha \in \Gamma$, $x(\alpha) \in B_\alpha$. Instead of $x(\alpha)$ we will also write x_α. The Boolean operations on $\underset{\alpha \in \Gamma}{\times} B_\alpha$ are defined coordinatewise. The projection maps of B_∞ onto each B_α will be denoted by π_α. Thus $\pi_\alpha(x) = x_\alpha$ for every $x \in \underset{\alpha \in \Gamma}{\times} B_\alpha$ and for every $\alpha \in \Gamma$. Now the inverse limit algebra B_∞ is defined as follows. $B_\infty = \{x : x \in \underset{\alpha \in \Gamma}{\times} B_\alpha, x_\alpha = f_{\alpha\beta} x_\beta$ whenever $\alpha \leqslant \beta, \alpha, \beta \in \Gamma\}$. It is easy to check that B_∞ is a subalgebra of $\underset{\alpha \in \Gamma}{\times} B_\beta$. The restrictions of the maps π_α to B_∞ will also be denoted by π_α.

We will designate the dual spaces of the B_α by X_α and the dual space of B_∞ by X. It is well-known that each X_α and X are zero dimensional compact Hausdorff spaces. Such spaces will also be called Boolean spaces. For every $x \in B_\infty$, x_α^* will denote the open-and-closed subset of X_α which corresponds with the element x_α of B_α. It follows from the representation theory of Boolean algebras and their dual spaces that with the *homomorphic* maps $f_{\alpha\beta} : B_\beta \to B_\alpha$, $\alpha \leqslant \beta$, there correspond *continuous* maps $f_{\alpha\beta}^* : X_\alpha \to X_\beta$, $\alpha \leqslant \beta$ such that the collection $\{X_\alpha, f_{\alpha\beta}^*\}$ becomes a *direct limit system* of Boolean spaces in the following sense: (1′) $f_{\alpha\alpha}^*$ is the identity map for every $\alpha \in \Gamma$; (2′) $f_{\alpha\gamma}^* = f_{\beta\gamma}^* f_{\alpha\beta}^*$ whenever $\alpha \leqslant \beta \leqslant \gamma$, $\alpha, \beta, \gamma \in \Gamma$.

Finally, the relation which exists between the maps $f_{\alpha\beta}$ and $f_{\alpha\beta}^*$ is given by

$$(f_{\alpha\beta}^*)^{-1}(x_\beta^*) = (f_{\alpha\beta} x_\beta)^* = x_\alpha^* \text{ for every } x \in B_\infty \text{ and } \alpha \leqslant \beta. \tag{3}$$

2. *Theorems on B_∞ and its dual space X.*

In this section we will prove some theorems which will be useful in the sequel.

Theorem 1. Let $\{B_\alpha, f_{\alpha\beta}\}$ be an inverse limit system of Boolean algebras over a directed set Γ whose inverse limit is B_∞. Let C be an arbitrary Boolean algebra such that for every $\alpha \in \Gamma$, $g_\alpha : C \to B_\alpha$ is a homorphic map such that

$$f_{\alpha\beta} g_\beta = g_\alpha, \text{ whenever } \alpha \leqslant \beta\ ,\ \alpha, \beta \in \Gamma. \tag{4}$$

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[2]) i.e. Γ is a partially ordered set such that for every α and $\beta \in \Gamma$ there exists a $\gamma \in \Gamma$, $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$.

Then there exists a unique homomorphic map $g : C \to B_\infty$ such that

(5) $$\pi_\alpha g = g_\alpha \text{ for all } \alpha \in \Gamma.$$

Proof. Let $y \in C$. Define $g(y)$ by $\pi_\alpha(g(y)) = g_\alpha(y)$. It is immediate from this definition that condition (5) is satisfied and it is also easy to check that g is a homomorphic map of C to $\underset{\alpha \in \Gamma}{\times} B_\alpha$. We proceed to show that $g(y) \in B_\infty$ for every $y \in C$. Suppose $\alpha \leqslant \beta$, then we have because of (4) that $g_\alpha(y) = f_{\alpha\beta} g_\beta(y)$ or $\pi_\alpha(g(y)) = f_{\alpha\beta}(\pi_\beta(g(y)))$ and thus $(g(y))_\alpha = f_{\alpha\beta}(g(y))_\beta$. But this implies that $g(y) \in B_\infty$. Finally, we must show that g is unique. Suppose g and g' would both satisfy (5). But then we would have for every $y \in C$, $\pi_\alpha g(y) = g_\alpha(y) = \pi_\alpha g'(y)$. It follows that $g(y) = g'(y)$ and hence $g = g'$.

The following theorem follows immediately as the dual of theorem 1.

Theorem 2. Let $\{X_\alpha, f^*_{\alpha\beta}\}$ be the direct limit system of Boolean spaces over the directed set Γ which corresponds with the inverse limit system $\{B_\alpha, f_{\alpha\beta}\}$ of Boolean algebras over the directed set Γ. Let X denote the dual space of B_∞ and let for every $\alpha \in \Gamma$, π^*_α be the continuous map of X_α to X which corresponds with π_α. Let Y be an arbitrary Boolean space such that for every $\alpha \in \Gamma$, $g^*_\alpha : X_\alpha \to Y$ is a continuous map such that

(4′) $$g^*_\beta f^*_{\alpha\beta} = g^*_\alpha, \text{ whenever } \alpha \leqslant \beta \;,\; \alpha, \beta \in \Gamma.$$

Then there exists a unique continuous map $g^* : X \to Y$ such that

(5′) $$g^* \pi^*_\alpha = g^*_\alpha \text{ for all } \alpha \in \Gamma.$$

3. *Representation of B_∞ as a field of subsets of the direct limit set of the X_α.*

We consider the direct limit system $\{X_\alpha, f_{\alpha\beta}\}$. The direct limit set X^∞ is defined as follows. Let Y be the union of the sets X_α, $\alpha \in \Gamma$. We define on Y a binary relation [2] by $p_\alpha \equiv q_\beta$, $\alpha, \beta \in \Gamma$, $p_\alpha \in X_\alpha$, $q_\beta \in X_\beta$, if and only if there exists a $\gamma \in \Gamma$, $\gamma \geqslant \alpha, \beta$, such that $f^*_{\alpha\gamma}(p_\alpha) = f^*_{\beta\gamma}(q_\beta)$. It is easy to check that this relation is an equivalence relation. Then X^∞ is by definition the set of equivalence classes of Y. We will denote the map which maps each element of Y onto the element of X^∞ to which it belongs by h. From now on we will also let Y stand for the topological sum of the spaces X_α, $\alpha \in \Gamma$. We note that Y is zero dimensional and Hausdorff but Y will in general not be compact. It is easy to see that the map $x \to x^* = \bigcup_{\alpha \in \Gamma} x^*_\alpha$, $x \in \underset{\alpha \in \Gamma}{\times} B_\alpha$, is an isomorphic map of $\underset{\alpha \in \Gamma}{\times} B_\alpha$ onto the field of all open-and-closed subsets of Y (cf. [1]). This field is reduced (i.e. any two points of Y lie in disjoint open-and-closed sets). It is clear that B_∞ is isomorphic to a subfield of this field since B_∞ is a subalgebra of $\underset{\alpha \in \Gamma}{\times} B_\alpha$. It has been proved in [1] that the dual space of $\underset{\alpha \in \Gamma}{\times} B_\alpha$ is the Stone–Cech compactification of Y. However we will not use this fact in this paper. We observe that as yet, we have not yet assigned a topology to X^{∞}. In order to do this we will first prove that B_{∞} is isomorphic to a field of subsets of X^{∞}.

Theorem 3. The map $x \to x^{**}$ defined by $x^{**} = h(x^{*})$ for every $x \in B_{\infty}$ is an isomorphic map of B_{∞} onto a field of subsets of X^{∞}.

Proof. Suppose x and y are elements of B_{∞} then it is immediate that $(x+y)^{**} = h(x+y)^{*} = h(x^{*} \cup y^{*}) = h(x^{*}) \cup h(y^{*}) = x^{**} \cup y^{**}$. Again, suppose that $x \in B_{\infty}$ then we will show that $\bar{x}^{**} = (x^{**})'$ (where $'$ denotes set-theoretic complement). Thus we must show that $x^{**} \cap \bar{x}^{**} = \emptyset$ and $x^{**} \cup \bar{x}^{**} = X^{\infty}$. Suppose p would be a point of X^{∞} such that $p \in x^{**} \cap \bar{x}^{**}$. Since $p \in x^{**}$ there exists an $\alpha \in \Gamma$ and a point $p_{\alpha} \in x_{\alpha}^{*}$ such that $h(p_{\alpha}) = p$. Similarly, since $p \in \bar{x}^{**}$ there exists a $\beta \in \Gamma$ and a point $p_{\beta} \in \bar{x}_{\beta}^{*}$ such that $h(p_{\beta}) = p$. It follows that $h(p_{\alpha}) = h(p_{\beta})$ and therefore there exists a $\gamma \in \Gamma, \gamma \geqslant \alpha, \beta$ such that $f_{\alpha\gamma}^{*}(p_{\alpha}) = f_{\beta\gamma}^{*}(p_{\beta})$. Now according to (3) we have $(f_{\alpha\gamma}^{*})^{-1}(x_{\gamma}^{*}) = x_{\alpha}^{*}$. But $p_{\alpha} \in x_{\alpha}^{*}$ thus $f_{\alpha\gamma}^{*}(p_{\alpha}) \in x_{\gamma}^{*}$. It follows that $f_{\beta\gamma}^{*}(p_{\beta}) \in x_{\gamma}^{*}$. But again, according to (3) we have $(f_{\beta\gamma}^{*})^{-1}(x_{\gamma}^{*}) = x_{\beta}^{*}$ and thus $p_{\beta} \in x_{\beta}^{*}$ but $p_{\beta} \in \bar{x}_{\beta}^{*}$. It follows that $x^{**} \cap \bar{x}^{**} = \emptyset$. Now suppose $p \in X^{\infty}, p \notin x^{**}$. We will show that $p \in \bar{x}^{**}$. Since $p \in X^{\infty}$ there exists an $\alpha \in \Gamma$ and a $p_{\alpha} \in X_{\alpha}$ such that $p = h(p_{\alpha})$. Now $p_{\alpha} \notin x_{\alpha}^{*}$, otherwise $p_{\alpha} \in X^{*}$ and then $p = h(p_{\alpha}) \in h(x^{*}) = x^{**}$. Since $p_{\alpha} \notin x_{\alpha}^{*}$ we have that $p_{\alpha} \in \bar{x}_{\alpha}^{*}$ and thus $p \in \bar{x}^{**}$. It follows that $x^{**} \cup \bar{x}^{**} = X^{\infty}$. We conclude from the previous discussion that the map $x \to x^{**}$ is a homomorphic map of B_{∞} onto a field of subsets of X^{∞}. It remains to show that it is an isomorphic map. Suppose $x^{**} = \emptyset$ for some $x \in B_{\infty}$. Then $h(x^{*}) = \emptyset$ and it follows that $x^{*} = \emptyset$ and thus $x = 0$. This completes the proof of the theorem.

4. *Topological structure of* X^{∞}.

We have seen in theorem 3 that $\{x^{**}, x \in B_{\infty}\}$ is a field of subsets of X^{∞} which is isomorphic to B_{∞}. It seems natural to assign a topology to X^{∞} which is obtained by taking as a base for the open sets this field of sets. It is immediate that X^{∞} becomes a zero dimensional space. Furthermore the map $h : Y \to X^{\infty}$ becomes a continuous map. Indeed, for every basic set x^{**}, $x \in B_{\infty}$, we have $h^{-1}(x^{**}) = h^{-1}(h(x^{*})) = x^{*}$. However the space X^{∞} need not be a Hausdorff space. We will show in section 7 that if all the π_{α} are "onto", then X^{∞} is also Hausdorff (theorem 6). Although this restriction is actually not a very essential one, we will pursue the discussion of the general case first.

5. *The space* X_1^{∞}.

Let X_1^{∞} be the space obtained by identifying any two points p and q of X^{∞} which have the property that if $p \in x^{**}$ then $q \in x^{**}$, for all $x \in B_{\infty}$. Thus X_1^{∞} has the quotient topology and the corresponding continuous map of X^{∞} onto X_1^{∞} will be denoted by h_1. Furthermore, for every $x \in B_{\infty}$, x^{***} will also stand for $h_1(x^{**})$.

Theorem 4. The space X_1^∞ is a zero dimensional Hausdorff space and the family $\{x^{***}, x \in B_\infty\}$ is a field of open-and-closed subsets of X_1^∞ which is a base for the open subsets of X_1^∞ and which is isomorphic to B_∞.

Proof. A simple and straightforward argument shows that the map $x^{**} \to h(x^{**}) = x^{***}$ is an isomorphic map. Hence, according to theorem 3, the map $x \to x^{***}$ is an isomorphic map. Thus the family of sets $\{x^{***}, x \in B_\infty\}$ is a field of sets which is isomorphic to B_∞. Again, since $(h_1)^{-1}(x^{***}) = x^{**}$ for every $x \in B_\infty$, it follows that x^{***} is an open-and-closed subset of X_1^∞. If U is an open set of X_1^∞ then $(h_1)^{-1}(U)$ is a union of sets x^{**}, thus U is a union of sets x^{***}. Thus the family $\{x^{***}, x \in B_\infty\}$ is a base for the open stets of X_1. Finally, it is immediate that X_1^∞ is Hausdorff.

6. *The space X.*

In this section we will prove that the dual space X of B_∞ is the largest zero dimensional Hausdorff compactification of X_1^∞. Before we state this theorem we will first recall some properties of the family of zero dimensional Hausdorff compactifications of a zero dimensional Hausdorff space. Suppose B is a Boolean algebra and suppose that F is a reduced field of subsets of a set $\mathscr{F}$, such that F is isomorphic to B. Taking F as a base for the open sets of a topology, $\mathscr{F}$ can be made into a zero dimensional Hausdorff space which we will also denote by $\mathscr{F}$. It is well-known [1] that $\mathscr{F}$ cam be imbedded as a dense subspace of the dual space $S(B)$ of B. This imbedding has the following property. Let U be any open-and-closed subset of $S(B)$ then $U \cap \mathscr{F}$ is precisely that element of F which corresponds with the same element of B as U. Again, if $\mathscr{F}$ is a zero dimensional Hausdorff space and if F is a field of subsets of $\mathscr{F}$ such that F is a base for the open sets of $\mathscr{F}$ (thus the elements of F are open-and-closed subsets of $\mathscr{F}$) then the dual space of F is a zero dimensional Hausdorff compactification of $\mathscr{F}$. Finally, the family of zero dimensional Hausdorff compactifications of $\mathscr{F}$ has a largest one $\mathscr{F}^*$ in the following sense. Suppoze Z is an arbitrary zero dimensional compact Hausdorff space and suppose that $f : \mathscr{F} \to Z$ is a continuousmap, then f can be extended to a continuous map $f^* : \mathscr{F}^* \to Z$.

We have seen that the field $\{x^{***}, x \in B_\infty\}$ is a field of open-and-closed subsets of X_1^∞. It follows from the previous discussion that the dual space X of B is a zero dimensional Hausdorff compactification of X_1^∞ and we claim that it is the largest zero dimensional Hausdorff compactification. This statement, together with the results obtained in the previous sections, can now be summarized in the following theorem.

Theorem 5. Let $\{B_{\alpha\beta}, f_{\alpha\beta}\}$ be an inverse limit system of Boolean algebras over a directed set Γ whose inverse limit algebra is B_∞. Let $\{X_\alpha, f^*_{\alpha\beta}\}$ be the direct limit system of dual spaces and dual maps whose direct limit set is X^∞. Then X^∞ can be topologized in a natural way such that X^∞ becomes a zerodimensional Hausdorff space. Let X_1^∞ be the quotientspace of X^∞ which is obtained by identifying any two points of X^∞ which can not be separated by open subsets of X^∞. Then X_1^∞ is a zero dimensional space and the dual space X of B_∞ is the largest zero dimensional Hausdorff compactification of X_1^∞.

We only need to prove that X is the largest zero dimensional Hausdorff compactification of X_1^∞ and for this, we first need a lemma. We recall (section 1) that for every $\alpha \in \Gamma$, π_α denotes the projection map of B_∞ to B, and that π_α^* denotes the corresponding continuous map of X_α to X (theorem 2). Furthermore we have designated the natural ($=$continuous) map of the topological sum Y of the X_α onto X by h (section 3) and the quotient map of X^∞ onto X_1^∞ by h_1. Now let for every $\alpha \in \Gamma$, h_α denote the restriction of h to X_α and let $h_{1,\alpha}$ denote the restriction of h_1 to $h_\alpha(X_\alpha)$.

Lemma. For every $\alpha \in \Gamma$ we have $\pi_\alpha^* = h_{1,\alpha} h_\alpha$.

Proof. $h_{1,\alpha} h_\alpha$ is a continuous map of X_α to X_1^∞ and thus to X. Therefore $h_{1,\alpha}(h_\alpha(X_\alpha))$ is a closed subspace of X and thus there is a homomorphic map of B_∞ to B_α which corresponds with $h_{1,\alpha} h_\alpha$. In order to show that this homomorphic map is π_α we need to show that for every $x \in B_\infty$, we have $(h_{1,\alpha} h_\alpha)^{-1}(x^{***}) = (\pi_\alpha(x))^* = x_\alpha^*$.
Now

$$(h_{1,\alpha} h_\alpha)^{-1}(x^{***}) = h_\alpha^{-1}(h_{1,\alpha}^{-1}(x^{***})) = h_\alpha^{-1}(h_1^{-1}(x^{***}) \cap h(X_\alpha)) =$$
$$= h_\alpha^{-1}(x^{**} \cap h(X_\alpha)) = h^{-1}(x^{**}) \cap X_\alpha = x_\alpha^*.$$

Proof of theorem 5. We have already observed that we only need to prove that X is the largest zero dimensional Hausdorff compactification of X_1^∞. Let Z be an arbitrary zero dimensional compact Hausdorff space and let τ be a continuous map of X_1^∞ to Z, then we must show that τ can be extended to all of X. Let for every $\alpha \in \Gamma$, $g_\alpha^* : X_\alpha \to Z$ be defined by $g_\alpha^* = \tau\pi_\alpha^*$. We claim that $g_\beta^* f_{\alpha\beta}^* = g_\alpha^*$ for every $\alpha, \beta \in \Gamma$, $\alpha \leqslant \beta$. Indeed let p_α be a point of X_α then $g_\beta^*(f_{\alpha\beta}^*(p_\alpha)) = \tau(\pi_\beta^*(f_{\alpha\beta}^*(p_\alpha)))$. But $f_{\alpha\beta}^*(p_\alpha)$ and p_α lie in the same equivalence class and thus $h_\beta(f_{\alpha\beta}^*(p_\alpha)) = h_\alpha(p_\alpha)$ and thus $h_{1,\beta}(h_\beta(f_{\alpha\beta}^*(p_\alpha))) = h_{1,\alpha}(h_\alpha(p_\alpha))$ or $\pi_\beta^*(f_{\alpha\beta}^*(p_\alpha)) = \pi_\alpha^*(p_\alpha)$. It follows that $g_\beta^*(f_{\alpha\beta}^*(p_\alpha)) = \tau(\pi_\beta^*(f_{\alpha\beta}^*(p_\alpha))) = \tau(\pi_\alpha^*(p_\alpha)) = g_\alpha^*(p_\alpha)$. We infer from theorem 2 that there exists a continuous map $g^* : X \to Z$ such that $g^*\pi_\alpha^* = g_\alpha^*$ for every $\alpha \in \Gamma$. It remains to show that g coincides with τ on X_1^∞. Suppose p is a point of X_1^∞. Then there exists an $\alpha \in \Gamma$ and a point $p_\alpha \in X_\alpha$ such that $p = \pi_\alpha^*(p_\alpha)$ (We note that it is here that we use the lemma). Thus $g^*(p) = g^*(\pi_\alpha^*(p_\alpha)) = g_\alpha^*(p_\alpha) = \tau(\pi_\alpha^*(p_\alpha)) = \tau(p)$. This completes the proof of the theorem.

7. *A special case.*

Suppose that $\{B_\alpha, f_{\alpha\beta}\}$ is an inverse limit system of Boolean algebras over a directed set Γ whose inverse limit is B_∞. Now let for every

$\alpha \in \Gamma$, $B'_\alpha = \pi_\alpha B_\alpha$ and let for every $\alpha, \beta \in \Gamma$, $\alpha \leqslant \beta$, $f'_{\alpha\beta}$ be the restriction of $f_{\alpha\beta}$ to B'_β. Then it is easy to see that all the maps $f'_{\alpha\beta}$ are "onto" and that B_∞ is also the inverse limit of the inverse limit system $\{B'_{\alpha\beta}, f'_{\alpha\beta}\}$. It follows that if we impose on the inverse limit system $\{B_\alpha, f_{\alpha\beta}\}$ the extra condition [2] that all the maps $\pi_\alpha : B_\infty \to B_\alpha$ are "onto then this condition does not really affect the generality. We shall show that the topological consequence of this condition is, that the space X^∞ is a Hausdorff space. Thus we arrive at the following theorem.

Theorem 6. Let $\{B_\alpha, f_{\alpha\beta}\}$ be an inverse limit system of Boolean algebras over a directed set Γ whose inverse limit algebra is B_α. Suppose that all the maps $\pi_\alpha : B_\infty \to B_\alpha$ are "onto". Let $\{X_\alpha, f^*_{\alpha\beta}\}$ be the direct limit system of dual spaces and dual maps whose direct limit set is X^∞. Then X^∞ can be topologized in a natural way such that X^∞ becomes a zero dimensional Hausdorff space whose largest zero dimensional Hausdorff compactification is the dual space X of B_∞.

Proof. We only need to prove that, under the hypothesis of the theorem, the space X^∞ is Hausdorff. Let p and q be two distinct points of X^∞. There exist $\alpha, \beta \in \Gamma$ and points $p_\alpha \in X_\alpha$ and $q_\beta \in X_\beta$ such that $h(p_\alpha) = p$ and $h(q_\beta) = q$. Let $\gamma \geqslant \alpha, \beta$ then it is clear that $f^*_{\alpha\gamma}(p_\alpha) \neq f^*_{\beta\gamma}(q_\beta)$. Now $f^*_{\alpha\gamma}(p_\alpha)$ and $f^*_{\beta\gamma}(q_\beta)$ are points of the dual space X_γ of B_γ, and it follows from the fact that all the maps π_α, $\alpha \in \Gamma$, are "onto" that there exists an element $x \in B_\infty$ such that $f^*_{\alpha\gamma}(p_\alpha) \in x^*$ and $f^*_{\beta\gamma}(q_\beta) \in \bar{x}^*$. Now $p = h(p_\alpha) = h(f^*_{\alpha\gamma}(p_\alpha)) \in h(x^*) = x^{**}$, and $q = h(q_\beta) = h(f^*_{\beta\gamma}(q_\beta) \in h(\bar{x}^*) = \bar{x}^{**}$. It follows that X^∞ is Hausdorff.

Remark. The injection map of X^∞ into X is precisely the map which is used in [2] to show that there is a one-one map of X^∞ onto a subset of X. However the conclusion that this subset is always closed is incorrect. In fact, we have seen that X^∞ is always a dense subset of X and this subset is in general a proper subset of X (cf. section 8). We also note that if all the π_α are "onto" then the maps $f^*_{\alpha\beta}$ are one-one.

8. *Inverse limit sequences of Boolean algebras.*

Suppose that $B_0, B_1 \ldots, B_n, \ldots,$ is a sequence of Boolean algebra and suppose that for each $n \geqslant 1$ there exists a homomorphic map $f_n : B_n \to B_{n-1}$. Then the system $\{B_n, f_n\}$ can be considered as an inverse limit sequence of Boolean algebras if one defines for all n, $f_{nn} = 1$ and for $m > n$, $f_{nm} = f_{n+1}, \ldots, f_{m-1} f_m$. Again, we shall denote the inverse limit of this sequence by B_∞. With this inverse limit sequence there is associated a direct limit sequence of dual spaces and dual maps which again will be denoted by $\{X_n, f_n{}^*\}$. Finally, X^∞ will again stand for the direct limit set of this sequence and X for the dual space of B_∞. Now if we assume that all the maps f_n are "onto" then it is not difficult to see that all the projection maps $\pi_n : B_\infty \to B_n$ are "onto" and it follows from theorem 6 that X^∞

is a Hausdorff space under its natural topology. Moreover, all the maps f_n^* are one-one (thus each f_n^* is a homeomorphic map of X_{n-1} onto $f_n(X_{n-1})$. In addition, we will show that in this case X^∞ has the quotient topology relative to the topological sum Y of the X_n and to the natural map h of Y on X^∞. Thus we have the following theorem.

Theorem 7. Let $\{B_n, f_n\}$ be an inverse limit sequence of Boolean algebras whose inverse limit algebra is B_∞. Suppose that all the maps f_n are "onto". Let $\{X_n, f_n^*\}$ be the direct limit sequence of dual spaces and dual maps whose direct limit set is X^∞. Let Y be the topological sum of the spaces X_n and let h be the natural map of Y onto X^∞. If we assign to X^∞ the quotient topology relative to the space Y and to the map h, then the dual space X of B_∞ is the largest zero dimensional Hausdorff compactification of X^∞.

Proof. We only need to show that the topology of X^∞ which is obtained by taking the family $\{x^{**}=h(x^*), x \in B_\infty\}$ as a base for the open sets is the quotient topology relative to Y and h. Thus we must show that if U is an subset of X^∞ such that $h^{-1}(U)$ is open in Y then $h^{-1}(U)$ is the union of sets $x^*, x \in B_\infty$. Now suppose that p is a point of $h^{-1}(U)$ then $p \in X_n$ for some fixed n. Let I be the ideal of B_{n+1} which corresponds with the open set $h^{-1}(U) \cap X_{n+1}$. Now $f_{n+1}(I)$ is an ideal of B_n which corresponds with the open set $h^{-1}(U) \cap X_n$. But $x_n^* \in h^{-1}(U)$ thus $x_n \in f_{n+1}(I)$. It follows that there exists an element x_{n+1} of B_{n+1} such that $x_{n+1} \in I$ and such that $f_{n+1}(x_{n+1})=x_n$. Moreover we have that $x_{n+1}^* \in h_{-}^{1}(U)$. Proceeding by induction, we can find for every $k>n$ an element $x_k \in B_k$ such that $x_k^* \in h^{-1}(U)$ and such that $f_k(x_k)=x_{k-1}$. Now let for every $k<n$, x_k be the element of B_k which is defined by $x_k=f_{kn}(x_n)$ then again $x_k^* \in h^{-1}(U)$ for all $k<n$. Finally, let x be the element of the direct product of the Boolean algebras $B_0, B_1, \ldots,$ whose kth coordinate is the element x_k as defined above for all $k=1, 2, \ldots,$ then it is clear that $x \in B_\infty$ and $p \in x^* \subset h^{-1}(U)$. It follows that $h^{-1}(U)$ is a union of sets x^*, $x \in B_\infty$. This completes the proof of the theorem.

Example. Let for every $n=0, 1, 2, \ldots,$ B_n denote the Boolean algebra 2^n and let for every $n \geqslant 1$, f_n be a homomorphic map of B_n onto B_{n-1}. Then the system $\{B_n, f_n\}$ is an inverse limit system of Boolean algebras, whose inverse limit algebra B_∞ is the Boolean algebra $2^{\aleph_0}$. It is easy to see that X^∞ has the discrete topology and consists of $\aleph_0$ points. The dual space X of B_∞ is the largest zero dimesional Hausdorff compactification of the discrete space of $\aleph_0$ points. It is well-known that in this case this largest zero dimensional Hausdorff compactification coincides with the Stone–Čech compactification.

We want to close this section with the following remark. We have seen that if $\{B_\alpha, f_{\alpha\beta}\}$ is an inverse limit system of Boolean algebras such that all the maps π_α are "onto" then X^∞ can be topologized in such a way that X^∞ can be imbedded as a dense subspace of the dual space X of B_∞ (theorem 6). The question arises whether it can occur (except in trivial cases) that X^∞ is compact and thus that $X^\infty = X$. We will not answer this question in its full generality but we will restrict ourselve to a particular case which illustrates the general situation.

Let $\{B_n, f_n\}$ be an inverse limit sequence of Boolean algebras such that all f_n are "onto" and such that no f_n is a one-one map. We claim that in this case X^∞ is always a proper subset of X. We observe that if I is a prime ideal of B_∞ then either $\pi_n I$ is a prime ideal of B_n or it coincides with B_n for every n (cf. [2]). It follows that if I can be represented by a point of X^∞ then there exists an n such that $\pi_n I$ is proper. Thus in order to show that X^∞ is a proper subspace of X we must exhibit a prime ideal I of B such that $\pi_n I = B_n$ for all n. Since no f_n is a one-one map, there exists for every $n \geqslant 1$ an element $y_n \in B_n$, $y_n \neq 1$ such that $f_n(y) = 1$. Again, there exists for every $k = 0, 1$ $k = 1, 2, \ldots$ a sequence of elements $\{x_n^k, n = k+1, k+2, \ldots\}$ such that $x_n^k \in B_n$ and such that $f_{k+1}(x_{k+1}^k) = y_k$ and $f_n(x_n^k) = x_{n-1}^k y_{n-1}$ for $n \geqslant k+2$. Now let for every $k = 1, 2, \ldots, u^k$ be the element of the direct product of the $B_0, B_1, B_2, \ldots$ which is defined by $u_n^k = 1$ for $n < k$, $u_n^k = y_k$ for $n = k$ and $u_n^k = x_n^k y_n$ for $n > k$, where u_n^k denotes the nth coordinate of u^k for every k. Then it is not difficult to show that $u^k \in B_\infty$ for all k. Now observe that for every integer $p \geqslant 1$ $\sum_{k=1}^{p} u_k \neq 1$. Indeed the $(p+1)$th coordinate of $\sum_{k=1}^{p} u_k$ is clearly equal to $\sum_{k=1}^{p} x_{p+1}^k y_{p+1} = y_{p+1} \sum_{k=1}^{p} x_{p+1}^k \leqslant y_{p+1} < 1$. Thus $\sum_{k=1}^{p} u_k \neq 1$. Now let I^* be the ideal of B_∞ which is generated by the elements u^k, $k = 1, 2, \ldots$ then I^* is a proper ideal of B_∞ and thus I^* can be extended to a prime ideal I of B. But $u_n^{n+1} = 1$ for all $n = 1, 2, \ldots$ thus $\pi_n I_n = B_n$ for all n and it follows that I can not be represented by a point of X^∞.

Technological University of Delft
The Netherlands.

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